Incompressible flow as redistribution of accumulated difference: exact Navier–Stokes containment, conservative memory, and a finite ringing band

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Version 2 – 8 June 2026

Abstract

We study an incompressible continuum model built from an accumulated field $\mathbf{D}(\mathbf{x}, t)$ and its rate $\mathbf{P}(\mathbf{x}, t) = \partial_t \mathbf{D}$, interpreted as the active redistribution field. The pressure Π enforces $\nabla \cdot \mathbf{P} = 0$. The model contains incompressible Navier–Stokes exactly when the memory stiffness and rate friction are switched off, and otherwise adds two channels acting on different variables: a conservative restoring term $c^2 \Delta \mathbf{D}$ on the accumulated field and a linear friction term $-\lambda \mathbf{P}$ on its rate. We prove an exact energy balance showing that the restoring term stores and returns energy but does not dissipate it. Linearized transverse modes exhibit oscillatory relaxation if and only if $c^2 > \nu\lambda$, and then only on the finite wavenumber band

$$k \in (k_-, k_+), \qquad k_\pm = \frac{c \pm \sqrt{c^2 - \nu\lambda}}{\nu}.$$

Thus the simultaneous presence of a bulk friction channel and a gradient-viscous channel produces both a lower and an upper cutoff. The band edges, together with one measured ringing frequency, determine (ν, c, λ) by explicit inversion formulas. Finally, an exact modal phase diagnostic separates stored difference from active redistribution and yields the decay law $\dot{E}_k = -2\gamma_k E_k \sin^2 \theta_k$. The scalar core is validated numerically against conservation, monotone dissipation, the overdamped diffusion limit, and the predicted ringing band. In the small-deformation regime the accumulated field is displacement-like, placing the model within incompressible media with one elastic and two dissipative channels; the contribution is the explicit redistribution reading, the two-cutoff band, the inversion procedure, and the phase diagnostic.

Keywords: Navier–Stokes; incompressible flow; memory variables; viscoelasticity; modal ringing; energy identity.

1 Introduction and summary of results

The incompressible Navier–Stokes equations describe the evolution of a divergence-free velocity field and provide the standard mathematical model for Newtonian incompressible flow [1, 2, 3]. This manuscript studies a nearby question: what changes if the velocity is not taken as primitive, but instead as the *rate of an accumulated field*? We introduce a vector field $\mathbf{D}(\mathbf{x}, t)$ and identify its rate

$$\mathbf{P}(\mathbf{x}, t) = \partial_t \mathbf{D}(\mathbf{x}, t)$$

as the active redistribution field. Operationally, $\mathbf{D}$ is the time-integrated transverse redistribution field in the fixed-regime closure; depending on the medium, it may be interpreted as displacement-like, configurational, or coarse-grained. The incompressibility constraint $\nabla\cdot \mathbf{P} = 0$ says that redistribution reorganizes the accumulated field without local volumetric creation, while the pressure Π enforces the constraint.

The interpretation remains deliberately modest. The mathematics below does not require a microscopic ontology for $\mathbf{D}$; it requires only the stated variables, equations, and boundary conditions. The guiding reading is that *flow is incompressible redistribution of accumulated difference*, and the classical velocity description is recovered when the storage and friction channels are switched off.

The model, stated in Section 2, preserves the advective, pressure, and viscous structure of incompressible Navier–Stokes and adds exactly two terms: a restoring force $c^2\Delta\mathbf{D}$ acting on the *accumulated* field, and a linear friction $-\lambda\mathbf{P}$ acting on the *active* field. The results are:

R1 (*Exact containment; derived*). When $c = \lambda = 0$ the accumulated field decouples and the system is exactly incompressible Navier–Stokes with $\mathbf{P}$ as the velocity (Section 3).

R2 (*Conservative memory; derived*). The energy $E = \frac{1}{2}\int_\Omega(|\mathbf{P}|^2 + c^2|\nabla\mathbf{D}|^2)\,\mathrm{d}\mathbf{x}$ obeys an exact balance whose only dissipative contributions are $-\nu\|\nabla\mathbf{P}\|^2$ and $-\lambda\|\mathbf{P}\|^2$. The restoring term is conservative: it stores and returns energy, and supplies no smoothing (Section 4).

R3 (*Finite ringing band; prediction, untested*). Linearized modal relaxation is oscillatory iff $c^2 > \nu\lambda$, and then exactly for $|k| \in (k_-, k_+)$ with $k_\pm = \left(c \pm \sqrt{c^2 - \nu\lambda}\right)/\nu$; inside the band the mode rings at $\Omega_k = \sqrt{c^2k^2 - \gamma_k^2/4}$ under the envelope $e^{-\gamma_k t/2}$, $\gamma_k = \lambda + \nu k^2$ (Section 6).

R4 (*Parameter inversion; derived*). The band edges satisfy $k_- + k_+ = 2c/\nu$ and $k_-k_+ = \lambda/\nu$; together with one measured ringing frequency they determine (ν, c, λ) completely. The prediction is therefore self-calibrating: a measured band fixes the parameters, and the same parameters must then reproduce every other ringing frequency in the band (Section 7).

R5 (*Exact phase diagnostic; derived*). Per mode, an angle θ_k separates stored difference ($\theta_k \approx 0$) from active redistribution ($\theta_k \approx \pi/2$), with the exact decay law $\dot{E}_k = -2\gamma_k E_k \sin^2\theta_k$: *only the active component dissipates* (Section 8).

Three structural facts complete the picture. The vorticity formulation retains the vortex-stretching term, so the model does not remove the central three-dimensional difficulty (Section 9); the Helmholtz split of any coarse current shows that pressure absorbs exactly the longitudinal part, so that *the transverse part of the current is the flow* (Section 10); and in the small-deformation regime $\mathbf{D}$ is the displacement field, which locates the model honestly within the class of incompressible media carrying one elastic and two dissipative channels (Section 5). What the present formulation contributes to that class is the mechanism reading, the closed-form two-cutoff band with its inversion formulas, and the exact phase diagnostic. A numerical validation of the scalar core (Section 11) confirms every quantitative statement accessible at that level.

2 The model

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a periodic box or a smooth bounded domain with boundary conditions such that integrations by parts carry no boundary fluxes. The primitive fields are $\mathbf{D}(\mathbf{x}, t)$, $\mathbf{P}(\mathbf{x}, t)$, and $\Pi(\mathbf{x}, t)$. The fixed-regime system is

$$\begin{aligned} \partial_t \mathbf{D} &= \mathbf{P}, \\ \partial_t \mathbf{P} + (\mathbf{P}\cdot\nabla)\mathbf{P} &= -\nabla\Pi + \nu\Delta\mathbf{P} + c^2\Delta\mathbf{D} - \lambda\mathbf{P} + \mathbf{S}, \\ \nabla\cdot\mathbf{P} &= 0, \end{aligned} \tag{1}$$

with $\nu \geq 0$ a gradient (solvent-type) viscosity, $c \geq 0$ a restoring stiffness with the dimensions of a wave speed, $\lambda \geq 0$ a bulk friction rate, and $\mathbf{S}$ an external or unresolved forcing. The mechanism role of each term:

Term	Mechanism role
$\partial_t \mathbf{D} = \mathbf{P}$	active redistribution is the rate of the accumulated field
$(\mathbf{P}\cdot\nabla)\mathbf{P}$	redistribution transports itself
$-\nabla\Pi$	closure enforcing incompressibility
$\nu\Delta\mathbf{P}$	gradient dissipation of *active* redistribution
$c^2\Delta\mathbf{D}$	conservative restoration of *accumulated* difference
$-\lambda\mathbf{P}$	bulk friction on *active* redistribution
$\mathbf{S}$	forcing

The essential structural choice is that the elastic force acts on the accumulated field while both dissipative forces act on its rate. The two dissipative channels act on different objects and scale differently with wavenumber; this single fact generates the central prediction of Section 6.

A standing compatibility point is useful. Since $\nabla\cdot\mathbf{P} = 0$ and $\partial_t\mathbf{D} = \mathbf{P}$,

$$\partial_t(\nabla\cdot\mathbf{D}) = 0.$$

Thus the longitudinal part of $\mathbf{D}$ is fixed by the initial data. In the main statements we either assume $\nabla\cdot\mathbf{D}_0 = 0$ or, equivalently, regard any time-independent longitudinal component as a pressure-absorbed gauge that does not contribute to the transverse flow dynamics.

The local law $\partial_t\mathbf{D} = \mathbf{P}$ is a first closure: local, Eulerian, fixed-frame, valid for small-to-moderate deformation. It is the closure under which the exact energy identity of Section 4 holds, and it is the regime in which all results of this paper are stated. Finite-deformation transport of $\mathbf{D}$ (a convected or corotational law) is a well-posed extension, deferred because it modifies the energy bookkeeping; the results below are exact within the stated regime.

3 Exact containment of Navier–Stokes

Set $c = \lambda = 0$ in (1). The field $\mathbf{D}$ decouples from the $\mathbf{P}$-equation, which becomes

$$\partial_t\mathbf{P} + (\mathbf{P}\cdot\nabla)\mathbf{P} = -\nabla\Pi + \nu\Delta\mathbf{P} + \mathbf{S}, \qquad \nabla\cdot\mathbf{P} = 0 :$$

incompressible Navier–Stokes with $\mathbf{P}$ identified as the velocity field $\mathbf{u}$ [1, 2, 3]. The containment is exact, not asymptotic. Its reading is the central interpretive statement of the paper:

The classical incompressible flow equation is the memory-suppressed case of the redistribution of accumulated difference. The velocity description survives unchanged; what changes is what the velocity is the rate of.

The model does not derive viscosity, inertia, or pressure from anything deeper. It embeds the Navier–Stokes structure in a strictly larger system in which storage, restoration, and friction are explicit and individually switchable.

4 Energy identity: the memory channel is conservative

Define

$$E(t) = \frac{1}{2}\int_\Omega \left(|\mathbf{P}|^2 + c^2|\nabla\mathbf{D}|^2\right)\mathrm{d}\mathbf{x}.$$

Result 1 (Energy balance)**.** *For smooth solutions of* (1) *under periodic or compatible no-flux boundary conditions,*

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\nu\int_\Omega |\nabla\mathbf{P}|^2\,\mathrm{d}\mathbf{x} \;-\; \lambda\int_\Omega |\mathbf{P}|^2\,\mathrm{d}\mathbf{x} \;+\; \int_\Omega \mathbf{P}\cdot\mathbf{S}\,\mathrm{d}\mathbf{x}.$$

The derivation is given in Appendix A: the pressure term vanishes by incompressibility, the advective term by the standard cancellation, and the restoring term cancels exactly against the time derivative of $\frac{1}{2}c^2\|\nabla\mathbf{D}\|_2^2$. Three consequences are worth isolating.

First, $c^2\Delta\mathbf{D}$ is *not* a dissipative regularizer. It stores energy in gradients of the accumulated field and returns it. In particular the model should not be confused with rate-regularized or Voigt-type systems, where an added term acts on a rate variable and modifies the evolution operator [9, 10]: here the added elastic term acts on the accumulated history $\mathbf{D} = \int_0^t \mathbf{P}\,\mathrm{d}s$ and is energetically neutral. The distinction is structural, visible term-by-term in the balance.

Second, the two dissipative channels are non-redundant. The gradient channel removes energy at rate $\nu\|\nabla\mathbf{P}\|^2$ and acts on small scales; the friction channel removes it at rate $\lambda\|\mathbf{P}\|^2$ and acts on the bulk, including the gradient-free mode. Their distinct wavenumber scaling, νk^2 versus λ, is exactly what produces the finite band of Section 6.

Third, the identity is the bookkeeping law of the mechanism: every unit of energy is accounted for as active redistribution, stored difference, dissipation, or forcing. Nothing leaks and nothing appears.

5 Small-deformation identification

In the regime of the local law $\partial_t\mathbf{D} = \mathbf{P}$, the accumulated field is the time integral of the flow field at a fixed spatial point. For small deformations, this is naturally displacement-like. The fixed-regime model can therefore be read as an incompressible medium carrying one elastic channel, with stiffness c^2 acting on accumulated gradients, and two dissipative channels, namely a rate-gradient viscosity ν and a rate friction λ. This places the linear sector near classical viscoelastic and generalized-continuum descriptions, while keeping a different choice of primitive variable [5, 6, 7, 8].

This identification is intended as a grounding statement, not as a reduction of the proposal to an existing model. It shows that the equations are not exotic: their linear sector belongs to a physically familiar class of media. The present formulation contributes three operational features. First, it gives a mechanism reading in which the elastic channel restores accumulated difference and the constraint $\nabla\cdot\mathbf{P} = 0$ closes the transverse redistribution. Second, it isolates a closed-form two-cutoff ringing band and explicit inversion formulas (Sections 6–7). To our knowledge, this two-cutoff operational form has not been isolated explicitly in this setting.

Third, it provides the exact phase diagnostic of Section 8. The model's discriminating content is therefore structural: it counts dissipative channels, fixes their wavenumber scaling, and makes that count measurable from a single dispersion experiment.

6 The finite ringing band

Linearize (1) about rest and consider a single Fourier mode of wavenumber k (suppressing advection and pressure, exact for transverse shear modes). The modal amplitude obeys

$$\ddot{D}_k + \gamma_k \dot{D}_k + \varpi_k^2 D_k = 0, \qquad \gamma_k = \lambda + \nu k^2, \quad \varpi_k = c\,k. \tag{2}$$

The roots of $s^2 + \gamma_k s + \varpi_k^2 = 0$ separate two regimes: *underdamped* (oscillatory relaxation, ringing) for $\varpi_k > \gamma_k/2$ and *overdamped* (monotone relaxation) for $\varpi_k < \gamma_k/2$. Writing the underdamping condition $2ck > \lambda + \nu k^2$ as a quadratic in k gives:

Result 2 (Finite ringing band)**.** *For $\nu > 0$, oscillatory modal relaxation exists iff*

$$c^2 > \nu\lambda,$$

and then exactly for wavenumbers in the finite band $k \in (k_-, k_+)$ with

$$k_\pm = \frac{c \pm \sqrt{c^2 - \nu\lambda}}{\nu}.$$

Inside the band the mode rings at $\Omega_k = \sqrt{\varpi_k^2 - \gamma_k^2/4}$ under the envelope $e^{-\gamma_k t/2}$.

The limiting cases are part of the statement. If $\nu = 0$, $\lambda > 0$, the band degenerates to the half-line $k > \lambda/2c$: ringing with a lower cutoff only. If $\lambda = 0$, $\nu > 0$, the lower edge collapses and the band is $0 < k < 2c/\nu$: an upper cutoff only. If both vanish, every nonzero mode oscillates conservatively.

What the band measures. The band is a *channel counter.* A medium with no elastic channel ($c = 0$, the Navier–Stokes case) has an empty band: no ringing at any wavenumber. A medium with an elastic channel and a single dissipative channel constant in k rings above a lower cutoff with no upper edge: the band is a half-line. A medium with an elastic channel and a single dissipative channel growing as k^2 rings below an upper cutoff with no positive lower edge. Only the two-channel structure—one rate constant in k, one rate growing as νk^2—produces a band that is bounded on *both* sides, because at high wavenumber the gradient channel overwhelms the elastic frequency, $\gamma_k \gg \varpi_k$, and relaxation returns to monotone. The presence, absence, and shape of the band therefore read off the dissipative channel structure of the medium directly from linear response, with no fitting freedom beyond the three parameters themselves—and Section 7 shows that even those are fixed by the band.

7 Parameter inversion: the prediction is self-calibrating

The band edges are not merely qualitative markers. By Vieta's formulas applied to $\nu k^2 - 2ck + \lambda = 0$,

$$k_- + k_+ = \frac{2c}{\nu}, \qquad k_-\,k_+ = \frac{\lambda}{\nu}. \tag{3}$$

Measuring both edges of the band therefore fixes the two ratios c/ν and λ/ν. One additional absolute measurement—most cleanly, the ringing frequency Ω_{k^*} at any single wavenumber k^*

inside the band—then determines the overall scale, and hence all three parameters (ν, c, λ) separately:

$$\nu = \frac{2\,\Omega_{k^*}}{\sqrt{\left(k_- + k_+\right)^2 k^{*2} - \left(k_- k_+ + k^{*2}\right)^2}}, \qquad c = \frac{\nu\,(k_- + k_+)}{2}, \qquad \lambda = \nu\, k_- k_+.$$

(The expression for ν follows from $\Omega_{k^*}^2 = c^2 k^{*2} - \frac{1}{4}(\lambda + \nu k^{*2})^2$ after substituting (3).)

This closes the prediction operationally. An experiment that resolves the band produces the parameters with no free fitting; the same parameters must then reproduce the ringing frequency Ω_k and the decay envelope $e^{-\gamma_k t/2}$ at *every other* wavenumber in the band. Agreement across the measured band would support the model as a quantitative linear-response description; systematic disagreement would delimit its range of validity. The dimensionless control parameter of the entire linear sector is the single number

$$\Lambda = \frac{c^2}{\nu\lambda},$$

with ringing iff $\Lambda > 1$ and band width governed by $\sqrt{1 - \Lambda^{-1}}$.

8 Exact modal phase: stored versus active difference

For a single mode, set $X_k = \varpi_k D_k$ (stored component) and $Y_k = P_k = \dot{D}_k$ (active component), so that the modal energy is $E_k = \frac{1}{2}(X_k^2 + Y_k^2) = \frac{1}{2} R_k^2$ with

$$X_k = R_k \cos\theta_k, \qquad Y_k = R_k \sin\theta_k, \qquad \theta_k = \operatorname{atan2}(Y_k, X_k).$$

The unforced modal dynamics (2) becomes, exactly,

$$\dot{R}_k = -\gamma_k R_k \sin^2\theta_k, \qquad \dot{\theta}_k = -\varpi_k - \frac{\gamma_k}{2}\sin(2\theta_k), \qquad \frac{\mathrm{d}E_k}{\mathrm{d}t} = -2\gamma_k E_k \sin^2\theta_k \le 0. \qquad (4)$$

The decay law is the sharpest statement of the mechanism at modal level: *dissipation acts only on the active component.* When the difference is fully stored ($\theta_k = 0$) the mode loses no energy; when it is fully active ($\theta_k = \pi/2$) it dissipates at the maximal rate $2\gamma_k E_k$. The angle equation reproduces the band of Section 6 from a different direction: a stationary angle exists only when $\varpi_k/\gamma_k \le \frac{1}{2}$ (overdamped corner, the mode locks into a fixed storage/activity ratio and decays monotonically), while for $\varpi_k/\gamma_k > \frac{1}{2}$ no stationary angle exists, θ_k rotates, and the mode perpetually exchanges stored and active difference—which *is* the ringing. The diagnostic is exact mode-by-mode and approximate for broadband fields.

9 Vorticity

Let $\boldsymbol{\omega} = \nabla \times \mathbf{P}$ and $\boldsymbol{\Omega}_{\mathbf{D}} = \nabla \times \mathbf{D}$, so $\partial_t \boldsymbol{\Omega}_{\mathbf{D}} = \boldsymbol{\omega}$. Taking the curl of (1),

$$\partial_t \boldsymbol{\omega} + (\mathbf{P}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{P} + \nu\Delta\boldsymbol{\omega} + c^2\Delta\boldsymbol{\Omega}_{\mathbf{D}} - \lambda\boldsymbol{\omega} + \nabla \times \mathbf{S}.$$

The accumulated field acquires its own vorticity $\boldsymbol{\Omega}_{\mathbf{D}}$, restored conservatively by $c^2\Delta\boldsymbol{\Omega}_{\mathbf{D}}$ and fed by the active vorticity; the friction $-\lambda\boldsymbol{\omega}$ damps active vorticity uniformly across scales, including the largest. The vortex-stretching term $(\boldsymbol{\omega}\cdot\nabla)\mathbf{P}$ remains untouched: the fixed-regime model does not remove the central three-dimensional difficulty of the classical equations, and we make no regularity claim [4, 3]. Any smoothing must come from the dissipative parameters, not from the conservative memory channel—consistent with Result 1.

10 Pressure as absorption of the longitudinal current

The constraint $\nabla\cdot\mathbf{P}=0$ states mode-by-mode that $\mathbf{k}\cdot\mathbf{P}_{\mathbf{k}}=0$: redistribution is transverse. The constraint acquires content through the Helmholtz–Leray decomposition [1, 3]. If a coarse current $\mathbf{J}$ is constructed from any underlying description, it splits uniquely (under the stated boundary conditions) as

$$\mathbf{J}=\mathbf{P}+\nabla\psi, \qquad \nabla\cdot\mathbf{P}=0,$$

a transverse part and a longitudinal part. Taking the divergence of the $\mathbf{P}$-equation gives the pressure closure

$$\Delta\Pi=-\nabla\cdot\big[(\mathbf{P}\cdot\nabla)\mathbf{P}\big]+c^2\Delta(\nabla\cdot\mathbf{D})+\nabla\cdot\mathbf{S},$$

so the longitudinal component never appears as flow: it is absorbed, instantaneously, by the pressure response. The reading is a decomposition statement, valid in every regime:

> *The longitudinal part of the current is taken up by pressure; the transverse part is the flow.*

The Leray projection is meaningfully applied to $\mathbf{J}$ (not to $\mathbf{P}$, which is already transverse): $\mathbf{P}=\mathbb{P}_{\text{Leray}}\mathbf{J}$. Whether incompressibility itself can be derived from a deeper closure principle, rather than imposed, is left open; the present manuscript imposes it.

11 Scalar core: limits and numerical validation

Suppressing advection, pressure, and vector structure yields the scalar core

$$\partial_{tt}D-\nu\Delta\partial_t D+\lambda\,\partial_t D-c^2\Delta D=S, \tag{5}$$

whose corners organize the parameter space: a conservative wave ($\nu=\lambda=0$), a wave with linear drag ($\nu=0$), a wave with rate-gradient damping ($\lambda=0$), and the full inertial–dissipative case. No corner reduces to a textbook one-dimensional rheological element, because the elastic force acts on D while both dissipative forces act on its rate. When $\lambda>0$, the formal overdamped limit $\lambda\to\infty$ with $\mu=c^2/\lambda$ fixed gives fast relaxation $P\approx\mu\Delta D$ and hence the diffusion law

$$\partial_t D\approx\mu\,\Delta D.$$

In this limit only the ratio μ is observable; c^2 and λ are individually degenerate. Resolving them separately requires the inertial (wave) sector—precisely the sector in which the ringing band of Section 6 lives. The band is therefore not only the model's sharpest prediction but also its identifiability frontier.

Numerical validation. The $k=1$ mode on $[0,2\pi]$, $D=q(t)\sin x$, obeys $\ddot q+(\lambda+\nu)\dot q+c^2q=0$ with energy $E=\frac{1}{2}(\dot q^2+c^2q^2)$. Four checks were run with a high-order integrator (relative tolerance 10^{-12}):

1. *Conservation* ($\nu=\lambda=0$, $c=1$): relative energy drift below 5×10^{-12} over $t\in[0,40]$.
2. *Monotone dissipation* ($\nu=0.1$, $\lambda=0.4$, $c=1$): the energy is non-increasing to machine precision at every step, with final energy ratio 2.7×10^{-9} at $t=40$.
3. *Overdamped diffusion limit* ($\nu=0$, $\lambda=20$, $\mu=0.05$): after the initial fast layer the mode follows $q(t)=e^{-\mu k^2t}$ with relative error below 2.6×10^{-3}.

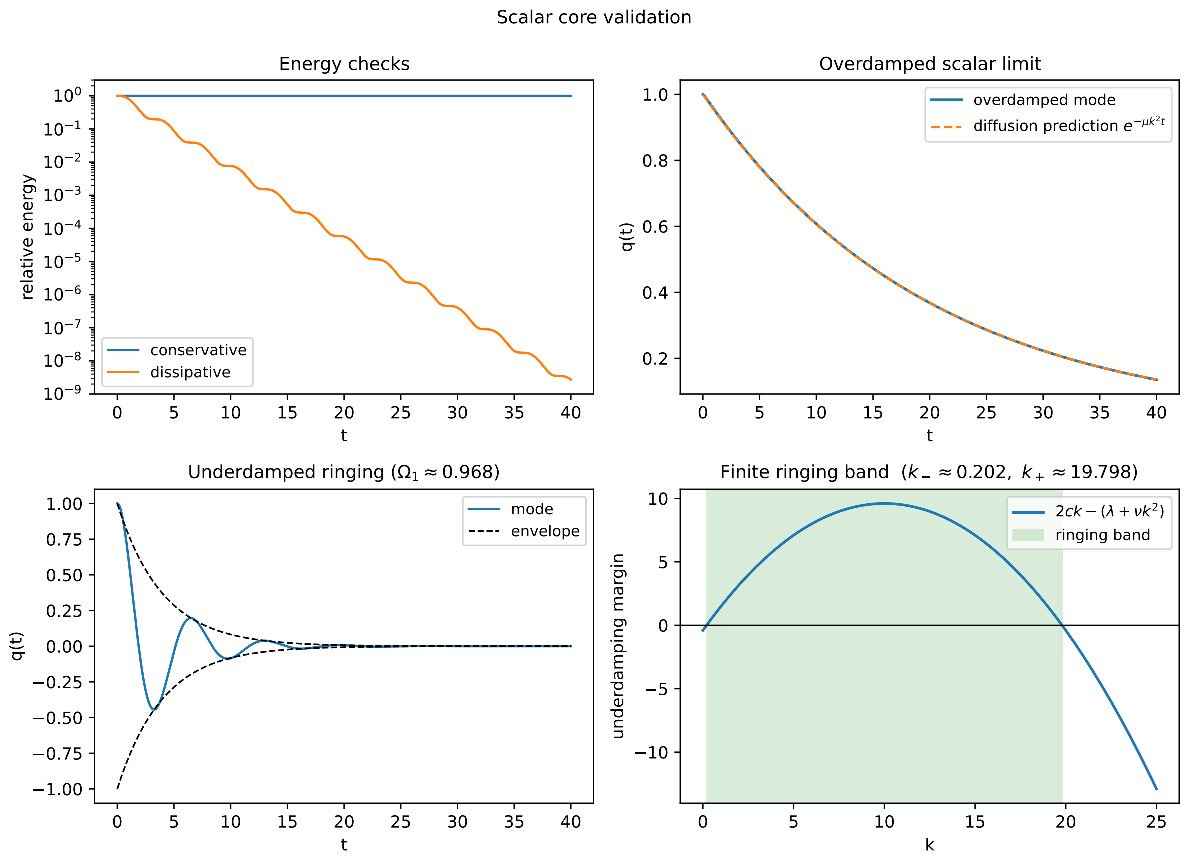

Figure 1: Scalar core validation. Top left: conservative energy is constant while dissipative energy decays monotonically. Top right: the overdamped mode follows the diffusion prediction $e^{-\mu k^2 t}$. Bottom left: the underdamped mode rings at the predicted Ω_1 under the damping envelope $e^{-\gamma_1 t/2}$. Bottom right: the underdamping margin $2ck - (\lambda + \nu k^2)$ is positive exactly on the predicted finite band (k_-, k_+).

4. *Ringing band* ($\nu = 0.1$, $\lambda = 0.4$, $c = 1$, so $\Lambda = c^2/\nu\lambda = 25 > 1$): the predicted band is $k_- \approx 0.2020$, $k_+ \approx 19.7980$; the $k = 1$ mode lies inside it and rings at the predicted $\Omega_1 = \sqrt{1 - (0.25)^2} \approx 0.9682$ under the envelope $e^{-0.25t}$. The inversion formulas (3) were verified numerically: $k_- + k_+ = 20.0000 = 2c/\nu$ and $k_- k_+ = 4.0000 = \lambda/\nu$.

These checks validate the scalar core completely: conservation, dissipation, the diffusion limit, the band location, the ringing frequency, and the inversion formulas all hold to the stated accuracy. The incompressible vector sector (Newtonian recovery at $c = \lambda = 0$, shear and Taylor–Green transients with $c, \lambda \neq 0$, energy identity in 2D/3D simulation) is the natural next computational stage.

12 Scope and outlook

Every result above is exact or numerically verified within its stated regime: local memory law, fixed material parameters, smooth solutions. Three extensions define the research frontier, each well-posed as a question. First, objective finite-deformation transport of $\mathbf{D}$, which must either preserve the energy balance of Result 1 or replace it with a controlled inequality. Second, the regime-transition mechanism: a fixed parameter set has finite capacity to absorb accumulated difference, and a threshold on $|\nabla \mathbf{D}|$ activating new material parameters is the natural next layer; it is constitutive until calibrated, and is therefore kept out of the core results. Third, the empirical test of the band itself: a calibrated medium in the two-channel class, probed in linear response across wavenumbers, either exhibits the two-cutoff band with the self-consistent parameters of Section 7 or it does not.

The standing of the proposal is then easy to state. As mathematics, it is a containment-plus-extension of the incompressible Navier–Stokes structure with one conservative and two dissipative channels, an exact energy balance, a closed-form dispersion prediction, and an exact modal diagnostic. As mechanism, it is the claim that the velocity field of classical fluid dynamics can be read, without loss, as the rate of an accumulated field—that flow is incompressible redistribution of stored difference, with pressure absorbing whatever part of the current is not flow. The first reading is fully established here. The second remains open by construction, and the model is built so that it can stay open: nothing in the equations requires deciding what is accumulated, only that something is.

A Energy identity

Take the L^2 inner product of the $\mathbf{P}$-equation in (1) with $\mathbf{P}$:

$$\begin{aligned}\int_\Omega \mathbf{P}\cdot\partial_t\mathbf{P}\,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{P}\cdot(\mathbf{P}\cdot\nabla)\mathbf{P}\,\mathrm{d}\mathbf{x}&\\ = -\int_\Omega \mathbf{P}\cdot\nabla\Pi\,\mathrm{d}\mathbf{x} + \nu\int_\Omega \mathbf{P}\cdot\Delta\mathbf{P}\,\mathrm{d}\mathbf{x} &+ c^2\int_\Omega \mathbf{P}\cdot\Delta\mathbf{D}\,\mathrm{d}\mathbf{x}\\ - \lambda\int_\Omega |\mathbf{P}|^2\,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{P}\cdot\mathbf{S}\,\mathrm{d}\mathbf{x}&.\end{aligned}$$

By incompressibility, $-\int \mathbf{P}\cdot\nabla\Pi = \int \Pi\,\nabla\cdot\mathbf{P} = 0$ and $\int \mathbf{P}\cdot(\mathbf{P}\cdot\nabla)\mathbf{P} = \frac{1}{2}\int(\mathbf{P}\cdot\nabla)|\mathbf{P}|^2 = 0$. Integration by parts gives $\nu\int \mathbf{P}\cdot\Delta\mathbf{P} = -\nu\int|\nabla\mathbf{P}|^2$, and, using $\mathbf{P} = \partial_t\mathbf{D}$,

$$c^2\int_\Omega \mathbf{P}\cdot\Delta\mathbf{D}\,\mathrm{d}\mathbf{x} = -c^2\int_\Omega \nabla\mathbf{P}:\nabla\mathbf{D}\,\mathrm{d}\mathbf{x} = -\frac{c^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\mathbf{D}|^2\,\mathrm{d}\mathbf{x}.$$

Collecting terms yields Result 1. The restoring contribution is a perfect time derivative: it moves energy into and out of storage and never destroys it.

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